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Two Results on Polynomial Interpolation in Equally Spaced Points

L. N. TREFETHEN*

Department of Computer Science, Cornell University, Ithaca, New York 14853, U.S.A.

AND

J. A. C. WEIDEMAN†

Department of Applied Mathematics, University of the Orange Free State, Bloemfontein 9300, South Africa

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We present two results that quantify the poor behavior of polynomial interpolation in n equally spaced points. First, in band-limited interpolation of complex exponential functions $e^{i\alpha x}$ ($\alpha \in \mathbb{R}$), the error decreases to 0 as $n \to \infty$ if and only if α is small enough to provide at least six points per wavelength. Second, the Lebesgue constant Λ_n (supremum norm of the nth interpolation operator) satisfies $\lim_{n \to \infty} \Lambda_n^{1/n} = 2$. Both of these results are more than 50 years old, but they are generally unknown to approximation theorists.

1. INTRODUCTION

It is well known that polynomial interpolation in equally spaced points can be troublesome—the "Runge phenomenon," discovered by Meray and Runge at the turn of the century. There is a standard result that quantifies this phenomenon: to ensure $p_n \to f$ in the supremum norm as $n \to \infty$, where p_n is the interpolant to a function f in $n+1$ equally spaced points on an interval, f must be analytic throughout a certain lens-shaped region of the complex plane [9, 10, 12, 22, 23, 37, 38]. By contrast, $p_n \to f$ is guaranteed for interpolation in Chebyshev points so long as f is somewhat smooth,

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† Current address: Department of Mathematics, Oregon State University, Corvallis, Oregon 97330.

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e.g., Lipschitz continuous. More precisely, it is sufficient for f to satisfy the Dini-Lipschitz condition $\omega(\delta)=o(|\log\delta|^{-1})$, where ω is the modulus of continuity [34, Theorem 14.4].

The purpose of this brief paper is to present two additional results on interpolation in equally spaced points which, although not new except in certain details, are generally unknown to approximation theorists.[1] Theorem 1 asserts that $\|p_n-f\|\to 0$ is guaranteed for equally spaced points, in a certain sense, if and only if f is appropriately band-limited: the grid must contain at least six points per wavelength. This theorem is implicit in the results of a paper by Carlson in 1915 [7], and generalizations can be found in the literature on entire functions [1, 2, 4, 25, 40]. Theorem 2 asserts that the Lebesgue constants Λ_n for equispaced interpolation grow asymptotically at a rate given by $\lim_{n\to\infty}\Lambda_n^{1/n}=2$. This result is due first to Turetskii in 1940 [35], then independently to Schönhage in 1961 [30], and an elementary proof was devised but not published by Jia in 1980 [20]. Some references to additional partial results, independent of Turetskii and Schönhage, are given in Section 4.

This paper proves both theorems by the use of the Newton interpolation formula. This method, besides providing simple proofs, reveals the natural parallel between the two. We are grateful to Christopher Budd for calling our attention to this technique [5].

In equispaced interpolation, and its applications discussed below, it is well known that rounding errors may render an algorithm useless even when it would perform successfully in exact arithmetic [12, 17, 32]. Theorem 1 suggests the explanation of this phenomenon that rounding errors, being essentially random, can hardly be expected to be band-limited. Theorem 2 quantifies their influence, predicting that floating-point computations with precision ε will generally be contaminated by errors of order $2^n\varepsilon$.

Figure 1 illustrates both of our theorems by a single set of numerical experiments carried out on a Sun Workstation with $\varepsilon=2^{-52}\approx 2.2\times 10^{-16}$. Each curve shows the computed sup-norm error in n-point interpolation of $\cos(\pi n x/\sigma)$ on $[-1,1]$ as a function of even integers n, where σ, taking values 5, 5.2, ..., 6.8, 7, represents the number of points per wavelength. The interpolants were computed stably by the barycentric formula described by Henrici [17]. Theorem 1, if not the precise constant six, can be seen in the fact that the errors for smaller σ apparently increase as $n\to\infty$, while those for larger σ decrease. Theorem 2 can be seen in the effects of rounding

[1] In fact, neither of the original papers [7, 35] is available in the Harvard or M. I. T. libraries. Nor, so far as we have been able to determine, is either of these papers—or the related paper of Schönhage [30]—cited in any books on approximation theory written in English.

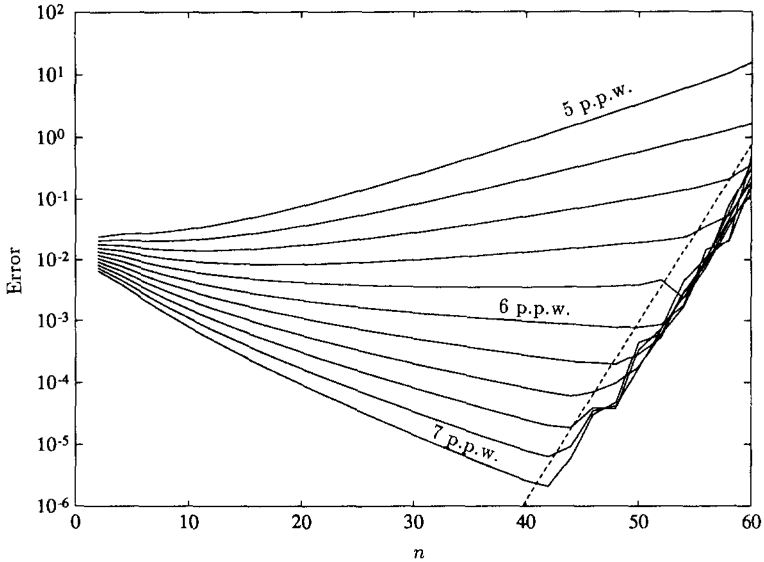

FIG. 1. Illustration of Theorems 1 and 2. Each curve shows sup-norm errors as a function of even integers n in n-point equispaced interpolation of $\cos(\pi n x/\sigma)$ on $[-1, 1]$ with a fixed number of points per wavelength, σ. The irregular results at the right are caused by rounding errors, and the dashes represent the curve $\varepsilon 2^{n+1}/en \log n \approx \varepsilon \Lambda_n$.

errors that corrupt the right edge of the figure. The dashed line represents ε times $2^{n+1}/en$ log n, Turetskii's more precise asymptotic estimate of Λ_n (see (4.8) below), and the fringe of rounding errors appears to parallel it closely.[2] Related figures are presented in [32].

Six points per wavelength is a critical number only if one is looking for accuracy precisely on the interval of interpolation; for a smaller interval such as $[-\frac{1}{2}, \frac{1}{2}]$, fewer points would suffice. Thus the Runge phenomenon is mainly a problem of behavior near endpoints, as has been recognized since the beginning of this subject.

Our interest in equispaced interpolation is motivated in part by the numerical solution of partial differential equations by spectral methods. These are collocation methods based on global polynomial interpolants; see [6]. In practice the Chebyshev points are the preferred collocation points, because of the superior convergence properties of the underlying interpolation process, as outlined above. It is generally accepted that collocation in equidistant points is not to be recommended. First, one

[2] One might ask why the rounding errors in Fig. 1 do not lie more nearly on top of the dashed line. The essential reason is that the Lebesgue constant, being defined as the norm of an operator, is a worst-case estimate. A complete explanation of the figure would require a backward error analysis of numerical interpolation by the barycentric formula.

might encounter non-convergence due to the Runge phenomenon, as was pointed out in relation to a two-point boundary value problem in [21, p. 586]. Second, due to the large Lebesque constants of Theorem 2, the process is extremely susceptible to rounding error; see [19] and [32].

A third and perhaps less obvious reason not to use equidistant points is related to our Theorem 1. Consider solving the heat equation $u_t = u_{xx}$ with Dirichlet boundary conditions. A spectral method based on a polynomial of degree $N+1$ leads to a semi-discrete system $v_t = Dv$, where v is a vector of approximate nodal function values and D is an $N \times N$ matrix representing the second derivative operator with Dirichlet conditions; see [6]. For stability purposes it is of interest to know the eigenvalue decomposition of D. The eigenfunctions of the continuous operator are of the form $u(x) = e^{i\alpha x}$; the eigenvectors of D correspond to accurate polynomial interpolants of these eigenfunctions, provided the interpolation process has sufficiently many points per wavelength to lead to convergence as $N \to \infty$. This means that, for α below some cut-off number, the eigenvalues and eigenvectors of D are accurate approximations of the eigenvalues and eigenfunctions of the continuous operator, but for α above this number they are not. Loosely speaking, if the eigenfunction oscillates too rapidly polynomial interpolation cannot resolve it, and a poor approximation of the eigenfunction and corresponding eigenvalue results. Theorem 1 asserts that this cut-off number in the case of equidistant interpolation corresponds to six points per wavelength. In practice this means that only one-third of the eigenvalues of D approximate those of the continuous operator accurately. The situation is much more favorable in the case of Chebyshev points: one requires only π points per wavelength for convergence (as discussed in Section 3), and a fraction of $2/\pi$ of the eigenvalues of D is accurate. We refer to [39] for more details and numerical verification of these results.

Interpolation in equally spaced points also underlies the derivation of Newton–Cotes formulas for numerical integration [22], and our theorems have natural analogs for this problem too.

2. The Newton Interpolation Formula Applied to $e^{i\alpha x}$

Let Δ denote the forward difference operator

$$\Delta f(x) = f(x+1) - f(x) \tag{2.1}$$

acting on functions defined on $\mathbb{R}$. Since $1 + \Delta$ is the forward shift operator, it is natural to associate the binomial series

$$(1+\Delta)^x = 1 + \binom{x}{1}\Delta + \binom{x}{2}\Delta^2 + \cdots \tag{2.2}$$

with a shift by an arbitrary distance x, but this series has only a formal meaning. Truncating it at the nth term, however, leads to polynomial interpolation,

$$p_n(x)=\left[1+\binom{x}{1}\Delta+\cdots+\binom{x}{n}\Delta^n\right]f(0), \tag{2.3}$$

where p_n is the unique polynomial of degree $\leqslant n$ that interpolates f at the points $0, 1, ..., n$. This result is known as the (equidistant) Newton or Newton–Gregory interpolation formula.

In particular, suppose f is the function

$$f(x)=e^{i\alpha x}, \qquad \alpha\in\mathbb{R}, \tag{2.4}$$

which implies

$$\Delta f=zf, \qquad z=(e^{i\alpha}-1). \tag{2.5}$$

Then $(1+\Delta)^x f(0)$ is the power series

$$(1+\Delta)^x f(0)=1+\binom{x}{1}z+\binom{x}{2}z^2+\cdots, \tag{2.6}$$

and $p_n(x)$ is the partial sum

$$p_n(x)=1+\binom{x}{1}z+\cdots+\binom{x}{n}z^n. \tag{2.7}$$

The values $z=z(\alpha)$ lie on the solid circle shown in Fig. 2.

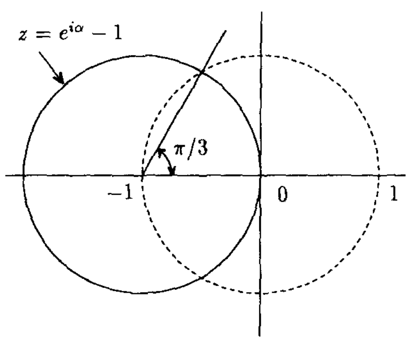

FIG. 2. Eigenvalues $z=e^{i\alpha}-1$ of the forward difference operator Δ.

3. Six Points per Wavelength

The idea behind Theorem 1 is that for (2.7) to be a good approximation to (2.6) we need $|z| \leqslant 1$, so that the series converges, and from Fig. 2 it is evident that this amounts to the condition $|\alpha| \leqslant \pi/3$: 6 points per wavelength. This is in contrast to the situation in trigonometric interpolation of periodic functions, where successful approximation requires only the well-known Nyquist sampling rate of 2 points per wavelength.

Theorem 1. *For each n, let* $p_n(x)$ *be the polynomial interpolant to* $f(x) = e^{i\alpha x}$, $\alpha \in \mathbb{R}$, *in the points* 0, 1, ..., n. *Then*

$$\|f - p_n\|_{[0,n]} \to 0 \qquad \text{as} \quad n \to \infty \tag{3.1}$$

if and only if $|\alpha| \leqslant \pi/3$.

In this statement $\|\cdot\|_{[0,n]}$ denotes the supremum norm on $[0, n]$. For the proof and the remainder of the paper, however, we shall write the norm simply as $\|\cdot\|$. Besides avoiding clutter, the reason is that the choice of the interval $[0, n]$ is just a convenience; one could equally well consider a fixed interval such as $[0, 1]$ or $[-1, 1]$, independent of n. Theorem 1 would then be stated in terms of a family of functions such as $f_n(x) = f(nx)$.

Proof. For any fixed value x, (2.6) is the Taylor series of the standard branch of the analytic function $(1+z)^x$, which we shall denote by $\phi(z)$. If x is a nonnegative integer, ϕ is entire and the series converges to $\phi(z)$ for all $z \in \mathbb{C}$ (trivially, since it reduces to a finite sum). If x is not a nonnegative integer, ϕ has an isolated singularity at $z = -1$, and the series converges to $\phi(z)$ for $|z| < 1$ and diverges for $|z| > 1$. On the circle $|z| = 1$, it converges to $\phi(z)$ for all $z \neq -1$. This follows from Theorem 5.4.5 and its corollary of Hille [18], which are based on Abel summation by parts, since the coefficients $\binom{x}{k}$ alternate in sign for $k > x$ and decrease in magnitude to zero as $k \to \infty$.

To derive (3.1), suppose first $|\alpha| \leqslant \pi/3$, which implies $|z| \leqslant 1$ and $z \neq -1$ (Fig. 2). Then for any $x \in \mathbb{R}$, $\phi(z) = f(x)$, so by the remarks above, (2.6) is convergent representation of $f(x)$. Subtracting (2.7) yields the series

$$f(x) - p_n(x) = \binom{x}{n+1} z^{n+1} + \binom{x}{n+2} z^{n+2} + \cdots, \tag{3.2}$$

which must consequently converge to 0 as $n \to \infty$. We need to show that a bound on $(f - p_n)(x)$ holds uniformly for $x \in [0, n]$, however, and this follows by considering the argument of Hille a little more carefully (see also his Eq. (5.1.20) and Theorem 5.1.8). To summarize the calculation, without discussing the details that imply convergence, let us define $a_k = (-1)^k \binom{x}{k}$. For $k \geqslant n \geqslant x$, these coefficients are all of the same sign and nonincreasing in magnitude. Therefore summation by parts converts (3.2) to

$$\begin{aligned} f(x) - p_n(x) &= \sum_{k=n+1}^{\infty} a_k(-z)^k \\ &= \sum_{k=n+1}^{\infty} (a_k - a_{k+1}) \sum_{j=n+1}^{k} (-z)^j \\ &= \sum_{k=n+1}^{\infty} (a_k - a_{k+1}) \frac{(-z)^{n+1} - (-z)^{k+1}}{1+z}, \end{aligned}$$

which implies

$$|f(x) - p_n(x)| \leqslant 2 \sum_{k=n+1}^{\infty} |a_k - a_{k+1}| = 2\,|a_{n+1}| = 2 \left| \binom{x}{n+1} \right| \tag{3.3}$$

for the values of z of interest (on the solid circular arc and inside the dashed circle in Fig. 2, with $|z| \leqslant 1$ and $|1+z| = 1$). Since $|\binom{x}{n+1}| \leqslant 1/(n+1)$ for $x \in [0, n]$ (readily proved by writing out $\binom{x}{n+1}$ explicitly), this quantity approaches 0 as $n \to \infty$, as claimed.

On the other hand, suppose $|\alpha| > \pi/3$. For some such values of α we have $|z| > 1$, in which case the terms of (2.7) are unbounded whenever x is not a nonnegative integer, so $f(x) - p_n(x)$ cannot converge to 0 or any other value. For other such values of α we have $|z| \leqslant 1$, but aliasing occurs. To be precise, $\phi(z)$ is now equal to $\tilde{f}(x)$ for some function $\tilde{f}(x) = e^{i(\alpha + 2\pi J)x}$, where J is a nonzero integer, since ϕ was defined as the standard branch of the Taylor series (2.6). The argument above now implies that $\|\tilde{f} - p_n\| \to 0$, and since $\|f - \tilde{f}\| \not\to 0$ as $n \to \infty$, $\|f - p_n\| \to 0$ is again ruled out. ∎

Theorem 1, though stated differently, is contained in results first proved by Carlson in 1915 [7, p. 53], and can be found in one form or another in various references on entire functions [1, 2, 4, 25, 40]. In particular, a natural generalization is to replace $i\alpha$ by an arbitrary complex number $w = u + iv$ in (2.4), so that the condition for convergence of (2.6) becomes

$$|e^w - 1| \leqslant 1,$$

which reduces to

$$u \leqslant \log(2\cos v) \qquad -\frac{\pi}{2} < v < \frac{\pi}{2}. \tag{3.4}$$

The region of values w represented by this condition is plotted in Fig. 4 of [2]. By considering superpositions of functions e^w with w in this region, one can establish convergence of polynomial interpolants in the sense of Theorem 1 for all functions in an appropriately defined subclass of the set of entire functions of exponential type. Since the smallest value $|w|$ on the border of the region (3.4) is log 2 (at $v=0$), this subclass includes all the entire functions of exponential type less than log 2. See, for example, Theorem 4.1 and Corollary 1 of [4] or Theorem 9.10.7 of [1].

In [39] a part of Theorem 1 is proved by means of the Hermite integral representation for the interpolation error. An argument like the one given here is presented in [5].

For interpolation of $f(x)=e^{i\alpha x}$ in Chebyshev rather than equally spaced points, or more generally for interpolation in the zeros or extrema of any Jacobi polynomial, it is shown in [39] that π points per wavelength on average are sufficient to ensure $\|f-p_n\| \to 0$; see also Theorem 3 of [13]. This amounts to 2 points per wavelength in the central, coarsest part of the grid.

We emphasize that the relevance of Theorem 1 to numerical calculations is limited severely by rounding errors, as illustrated already in Fig. 1.

4. Lebesgue Constants

The Lebesgue constant Λ_n is defined as the supremum norm of the interpolation operator:

$$\Lambda_n = \sup_{\|f\|=1} \|p_n\|. \tag{4.1}$$

Here, again, $\|\cdot\|$ denotes the supremum norm on $[0, n]$, but the results hold equally for interpolation on other intervals. One motivation for investigating Λ_n is that p_n satisfies the bound

$$\|f-p_n\| \leqslant (1+\Lambda_n)\|f-p_n^*\|, \tag{4.2}$$

where p_n^* is the polynomial of best approximation to f on $[0, n]$, and thus Λ_n quantifies how far from optimal an interpolant can be. For interpolation of a fixed function f on a fixed interval such as $[-1, 1]$, convergence can only be expected if f is smooth enough so that $\|f-p_n^*\|$ decreases as $n \to \infty$ faster than Λ_n increases. Another motivation, as discussed above, is that numerical interpolation in floating-point arithmetic will generally be useless, even for smooth functions f, whenever Λ_n is larger than the inverse of the machine precision ε.

We shall first state and prove Theorem 2, then relate it to results in the literature.

THEOREM 2. $\lim_{n\to\infty} \Lambda_n^{1/n} = 2$. *More precisely, for each integer* $n \geqslant 1$,

$$\frac{2^{n-2}}{n^2} < \Lambda_n < \frac{2^{n+3}}{n}. \tag{4.3}$$

Proof. Assume $n \geqslant 2$; the case $n = 1$ is immediate since $\Lambda_1 = 1$.

To prove the lower bound, let the polynomial interpolant to $f(x) = e^{i\pi x}$ be evaluated at $x = \frac{1}{2}$; by (2.7),

$$p_n\left(\frac{1}{2}\right) = 1 + \binom{1/2}{1}(-2) + \binom{1/2}{2}(-2)^2 + \cdots + \binom{1/2}{n}(-2)^n.$$

The first two terms cancel, and the remaining terms in the series are all negative, so the final term provides the required inequality:

$$\Lambda_n \geqslant \left| p_n\left(\frac{1}{2}\right)\right| \geqslant \left|\binom{1/2}{n}\right| 2^n = \left(\frac{\frac{1}{2}}{1}\frac{\frac{1}{2}}{2}\frac{\frac{3}{2}}{3}\frac{\frac{5}{2}}{4}\cdots\frac{n-\frac{3}{2}}{n}\right)2^n$$

$$= \frac{2^{n-2}}{n(n-1)}\left(\frac{\frac{3}{2}}{1}\frac{\frac{5}{2}}{2}\cdots\frac{n-\frac{3}{2}}{n-2}\right) \geqslant \frac{2^{n-2}}{n(n-1)} > \frac{2^{n-2}}{n^2}. \tag{4.4}$$

For the upper bound, note that for any f with $\|f\| \leqslant 1$ we have $\|\Delta f\| \leqslant 2$, and therefore (2.3) implies

$$|p_n(x)| \leqslant 1 + \left|\binom{x}{1}\right| 2 + \left|\binom{x}{2}\right| 2^2 + \cdots + \left|\binom{x}{n}\right| 2^n \tag{4.5}$$

for any x. By symmetry, it is enough to consider $x \leqslant n/2$. For such x, let us divide the series into two halves to obtain

$$|p_n(x)| \leqslant \left[1 + \left|\binom{x}{1}\right| 2 + \cdots + \left|\binom{x}{n/2}\right| 2^{n/2}\right]$$
$$+ \left[\left|\binom{x}{1+n/2}\right| 2^{1+n/2} + \cdots + \left|\binom{x}{n}\right| 2^n\right],$$

assuming for the moment that n is even. Since $|\binom{x}{k}| \leqslant \binom{n/2}{k}$ for $x, k \leqslant n/2$ (readily proved by writing out $\binom{x}{k}$ and $\binom{n/2}{k}$ explicitly and comparing terms), the first series is bounded by

$$1 + \binom{n/2}{1} 2 + \cdots + \binom{n/2}{n/2} 2^{n/2} = 3^{n/2}. \tag{4.6}$$

Since $|\binom{x}{k}| \leqslant 1/k$ for $k \geqslant x+1$ (a fact used already in the proof of Theorem 1), the second series is bounded by

$$\frac{2}{n}(2^{1+n/2}+\cdots+2^n)<\frac{2^{n+2}}{n}. \tag{4.7}$$

Since $3^{n/2}<2^{n+2}/n$ for $n \geqslant 2$, (4.6) and (4.7) combine to give the upper bound of (4.3) for even n. The case of odd n is similar. ∎

Now for the history, which is an unfortunate tale of duplicated efforts. To summarize, an asymptotically sharper result than Theorem 2 was published by Turetskii in 1940 [35], and independently by Schönhage in 1961 [30]:

$$\Lambda_n \sim \frac{2^{n+1}}{en\log n} \qquad \text{as} \quad n \to \infty. \tag{4.8}$$

Neither of these papers attracted much notice, however, at least in the West, and weaker results have been derived repeatedly by later authors. The following is a list of publications we have found that report bounds or asymptotic results on Λ_n; except as stated, none of them references any of the others.

- In 1912 Bernstein proved that equidistant interpolation of the function $|x|$ in the interval $[-1, 1]$ converges only at the points $x=-1, 0, 1$. In the proof, as reproduced in [24, p. 30], it is shown that $\lim_{n\to\infty} \Lambda_n = \infty$, but the precise growth rate is not established.
- In 1940 Turetskii proved (4.8) [35]; the result is repeated in his book of 1968 [36, Section 3.2].
- In 1961 Schönhage proved (4.8) again independently—in fact, a slightly stronger estimate with $\log n$ replaced by $\log n+\gamma$, where γ is Euler's constant [30]. The lower bound only is repeated in his book of 1971 [31, p. 126].
- In 1962 Golomb proved $\sqrt{e} \leqslant \text{lim-sup}_{n\to\infty}(\Lambda_n)^{1/n} \leqslant 2$ [16, Theorem 13.5].
- In 1969 Rivlin proved $C_1(\sqrt{3/2})^n \leqslant \Lambda_n \leqslant C_2(\sqrt{2}\,e)^n$ [27, Theorem 4.6]. Later, in 1974, Rivlin mentioned Golomb's bounds in a survey paper [29].
- In 1978 de Boor mentioned the lower bounds of both Golomb and Rivlin in [11]. The $\sqrt{e}$ lower bound is mentioned again in the textbook by Conte and de Boor [8].
- In 1980 Jia, having heard of Turetskii's result from de Boor,

devised a ten-line proof of $2^{n-2}/n(n-1) \leqslant \Lambda_n \leqslant 2^{n-1}$, but did not publish it [20].

- In 1982 Henrici derived a lower bound like that of (4.3) [17, p. 246].

Most recently, after seeing a preprint of the present paper, Fornberg has devised a partcularly elegant derivation of (4.8) [15]. His proof, like those in the papers cited above, is based on the Lagrange rather than the Newton form of the interpolating polynomial.

Undoubtedly there are other references that we are unaware of, to whose authors we apologize.

It is easy to pin down where our estimates leading to Theorem 2 have failed to be sharp. The lower bound can be improved by evaluating p_n at $x \approx 1/\log n$ instead of $x = \frac{1}{2}$ and by treating the third inequality in (4.4) with Stirling's formula. The upper bound can be improved by refining the estimate $|\binom{x}{k}| \leqslant 1/k$ before (4.7).

The numbers Λ_n can be calculated numerically by applying a minimization routine to find the maximum of the Lebesgue function in the interval [0, 1], and Fig. 3 compares results obtained in this way for $n \leqslant 100$ with the estimates of (4.3) and (4.8). Evidently Turetskii's formula is accurate even for small n. Tables of Λ_n for various $n \leqslant 50$ are given in [26] and [30].

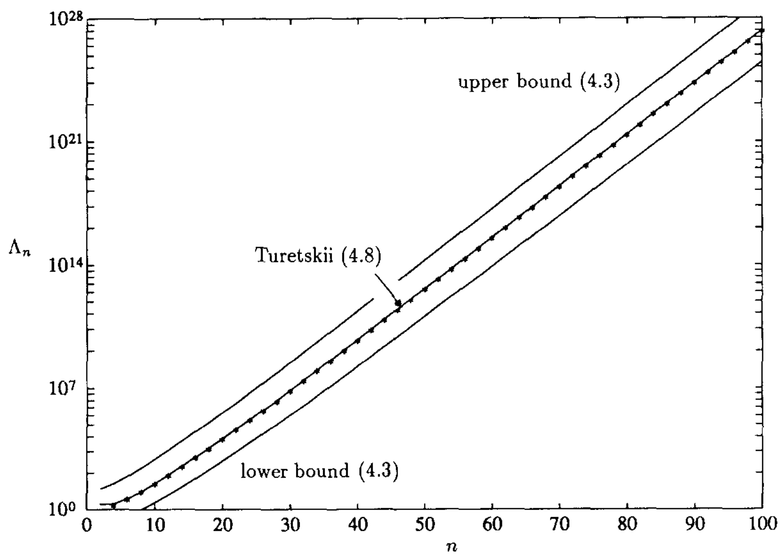

FIG. 3. Lebesgue constants Λ_n vs. n (log scale).

For interpolation in Chebyshev points on $[-1, 1]$, it is known that the Lebesgue constants grow at the much more favorable rate $\Lambda_n = (2/\pi) \log n + (2/\pi)(\gamma + \log(8/\pi)) + o(1)$ as $n \to \infty$, where γ is again Euler's constant [28, Theorem 1.2]. It is also known that *any* set of interpolation points leads to $\Lambda_n > (2/\pi) \log(n+1) + 1/2$ [14; 3, Eq. (41)], and thus the Chebyshev points are very nearly optimal. For interpolation in Legendre points, the growth of Λ_n is $O(\sqrt{n})$ as $n \to \infty$, but this figure falls to $O(\log n)$ again if one restricts attention to any subinterval $[-1+\varepsilon, 1-\varepsilon]$ [34, Section 14.4], or if one interpolates in the extreme points of the Legendre polynomials (together with ± 1) rather than their zeros [33].

In Eq. (6.2.4) of [22], Krylov states asymptotic formulas for the coefficients of Newton–Cotes quadrature formulas in the limit $n \to \infty$, due to R. Kuzmin, which reduce to

$$|c_{n/2}| \sim \frac{2^{n+3}}{\sqrt{2\pi}\, n^{5/2} \log^2 n} \qquad \text{as } n \to \infty \ (n \text{ even}) \tag{4.9}$$

for the largest (middle) coefficient of the n-point Newton–Cotes formula on $[0, 1]$. Since any error in the data at the middle grid point will be magnified by $|c_{n/2}|$, this result confirms that Newton–Cotes integration has essentially the same degree of ill behavior for large n as the interpolation process it is based upon.

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